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RELATIVELY CONVEX SUBSETS OF SIMPLY CONNECTED PLANAR SETS

BY

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ABSTRACT

Let $D \subset \Re^2$ be simply connected. A subset $K \subset D$ is **relatively convex** if $a, b \in K, [a, b] \subset D$ implies $[a, b] \subset K$. We establish the following version of Helly's Topological Theorem: If $\mathcal{K}$ is a family of (at least 3) compact, polygonally connected and relatively convex subsets of D, then $\bigcap \mathcal{K} \neq \emptyset$, provided each three members of $\mathcal{K}$ meet.

We also prove other results related to the combinatorial metric $\rho_K(a, b)$ (= smallest number of edges of a polygonal path from a to b in K).

1. Introduction

Let D be a subset of $\Re^2$. For points $a, b \in D$ we denote by $\rho_D(a, b)$ the smallest number of edges of a polygonal path in D that connect a and b. ($\rho_D(a, b) = \infty$ if there is no such path.) If D is polygonally connected, then ρ_D is an integer valued metric on D. This metric has particularly nice properties when D is simply connected. Such properties were first investigated by Bruckner and Bruckner in [BB]. It turns out that the notion of **relative convexity** is a particularly useful tool for dealing with these properties. (A subset K of D is **relatively convex** with respect to D if for every two points $x, y \in K$, $[x, y] \subset D$ implies $[x, y] \subset K$. Note that the intersection of any family of relatively convex subsets of D is again relatively convex.)

In this paper we establish the following results:

1. Let $\mathbf{P} = \langle a = p_0, p_1, \ldots, p_t = b \rangle$ ($t = \rho_D(a, b)$) be a ρ_D-minimal path from a to b in D. Note that $\mathbf{P}$ is necessarily simple (no self intersections). If

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x is a variable point that moves along $\mathbf{P}$ from a to b, then $\rho_D(a,x)$ is a monotone non-decreasing function of x (Lemma 2.8).

2. If K_1, K_2 are relatively convex and polygonally connected subsets of D with nonempty intersection, then $K_1 \cap K_2$ is also polygonally connected and relatively convex in D (Lemma 3.2).
   In fact, for points $a, b \in K_1 \cap K_2$, we determine the best possible upper bound for $\rho_{K_1 \cap K_2}(a,b)$ in terms of $\rho_{K_1}(a,b)$ and $\rho_{K_2}(a,b)$.
3. We give an elementary proof of Helly's Topological Theorem for compact, polygonally connected and relatively convex subsets of a simply connected set $D \subset \Re^2$. The theorem (Theorem 3.1) states that if D is a simply connected set in $\Re^2$, and $\{K_i : i \in I\}$ (with $|I| \geq 3$) is any family of subsets of D that are compact, polygonally connected and relatively convex in D, and such that each three have a point in common, then the intersection of the whole family is nonempty.

All these results will be used in a subsequent paper by the same authors [MP2] to establish a connection between the ρ-diameter and the ρ-radius of a compact simply connected planar set: *"If $K \subset \Re^2$ is compact and simply connected, and if $\rho_K(a,b) \leq n$ for all $a, b \in K$, then there is a point $z \in K$ such that $\rho_K(a,z) \leq \lceil \frac{n+1}{2} \rceil$ for all $a \in K$. This bound is best possible for all $n \geq 1$."*

2. Relatively convex subsets and the Monotonicity Lemma

2.1 Introductory results. We denote by $\operatorname{aff}(a,b)$ the line joining a and b, by $\mathrm{R}(a \to b)$ the ray issuing from a in the direction of b, and by $[a_1, \ldots, a_m]$ the convex hull of the points $a_1, \ldots, a_m$.

Definition 2.1: If $K \subset D$, the **star of K in D**, denoted by K' or $st(K,D)$, is the set of all points $y \in D$ that see a point of K via D. That is, $K' = \{y \in D : (\exists x \in K)([x,y] \subset D)\}$.

Definition 2.2: Let x be a point in D. The **n-th star of x in D** is the set $st_n(x,D) = \{z \in D :$ there exists a polygonal path of at most n edges joining x and z via $D\}$ $(= \{z \in D : \rho_D(x,z) \leq n\})$.

Observe that if $K = \{x\}$ then $K' = st_1(x,D) = st(x,D)$ is the usual star, and if $K = st_n(x,D)$, then $K' = st_{n+1}(x,D)$.

Lemma 2.3: *Let $\mathbf{Q} = \langle q_0, q_1, \ldots, q_n = q_0 \rangle$ be a simple closed polygon in $\Re^2$ $(n \geq 3)$. Denote by Q the union of $\mathbf{Q}$ and its interior. If t is an interior point of an edge $[q_i, q_{i+1}]$ of $\mathbf{Q}$, then t sees via Q some vertex of $\mathbf{Q}$ other than q_i, q_{i+1}.*

Proof: Consider a triangulation of Q by noncrossing interior diagonals of **Q**. (For the existence of such triangulations see [Hi].) In this triangulation, the boundary edge $[q_i, q_{i+1}]$ is an edge of a unique triangle $[q_i, q_{i+1}, q_j]$. Clearly, t sees q_j via Q. ∎

THEOREM 2.4: *If D is a simply connected set in $\Re^2$, and $K \subset D$ is relatively convex and polygonally connected, then K' is also relatively convex and polygonally connected.*

Proof: The fact that K' is polygonally connected is obvious. We proceed to show that K' is relatively convex in D. That is, suppose $a', b' \in K'$, $a' \neq b'$, $[a', b'] \subset D$. We will show that $[a', b'] \subset K'$. We distinguish the following cases:

CASE 1: a' sees (via D) a point $a \in K$ on $\text{aff}(a', b')$. In this case,

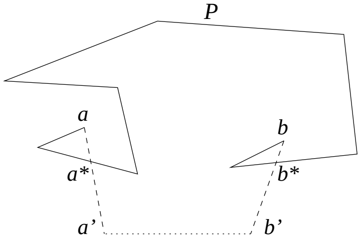
$$\text{conv}\{a, a', b'\} = [a, a'] \cup [a', b'] \subset D,$$

so every point on $[a', b']$ sees a via D, hence $[a', b'] \subset K'$.

CASE 2: b' sees (via D) a point $b \in K$ on $\text{aff}(a', b')$. This is exactly the same as Case 1.

CASE 3: a' and b' see (via D) a point $c \in K$ (not on $\text{aff}(a', b')$). In this case the boundary of the triangle $[a', b', c]$ is in D and therefore $[a', b', c] \subset D$, since D is simply connected. Thus every point on $[a', b']$ sees c via D, hence $[a', b'] \subset K'$.

CASE 4: Otherwise, assume a' sees (via D) a point $a \in K$, and b' sees (via D) a point $b \in K$, $a \neq b$, $a, b \notin \text{aff}[a', b']$. Let $P \subset K$ be a simple polygon that connects a with b. Let a^* be the first point on $[a', a]$ (going from a' to a) that belongs to P, and let b^* be the first point on $[b', b]$ (going from b' to b) that belongs to P (see Figure).



Let P^* be the subpolygon of P with endpoints a^*, b^*. By our construction $P^* \cap [a', a^*] = \{a^*\}$ and $P^* \cap [b', b^*] = \{b^*\}$. Moreover, $a^* \neq b^*$, $[a', a^*] \cap [a', b'] = \{a'\}$ and $[b', b^*] \cap [a', b'] = \{b'\}$ (otherwise we are in one of the previous cases).

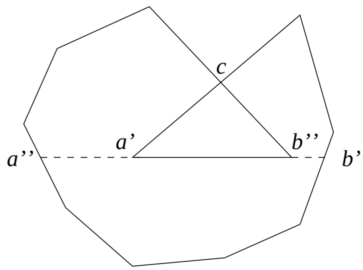
Also $P^* \subset K$ and $[a', b'] \cap K = \emptyset$ (see cases 1,2); hence $[a', b'] \cap P^* = \emptyset$. The segments $[a', a^*]$ and $[b', b^*]$ may be disjoint, or they may cross. (All other types of intersection of these two segments would lead us to one of the cases 1,2,3.)

Case 4.1: $[a', a^*] \cap [b', b^*] = \emptyset$. In this case $[a', a^*] \cup P^* \cup [b^*, b'] \cup [b', a']$ is a simple closed polygon in D, and its interior is in D as well, D being simply connected. By Lemma 2.3, every interior point of $[a', b']$ sees a point of $P^*(\subset K)$ via D, and thus $[a', b'] \subset K'$.

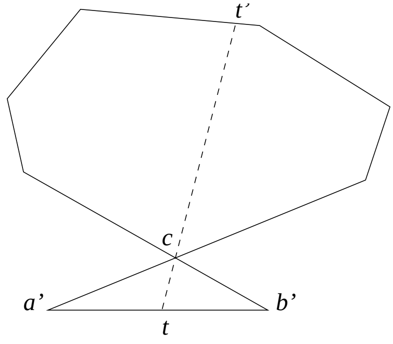
Case 4.2: $[a', a^*]$ and $[b', b^*]$ cross at a point c interior to both intervals.

Define $\mathbf{Q} = [c, a^*] \cup P^* \cup [b^*, c]$. $\mathbf{Q}$ is a simple closed polygon in D, and $[a', b'] \cap \mathbf{Q} = \emptyset$. If $[a', b']$ lies in the interior of $\mathbf{Q}$, let us extend the interval $[a', b']$ beyond a' until it meets $\mathbf{Q}$ at a point a'' and also beyond b' until it meets $\mathbf{Q}$ at a point b'' .

Clearly, a'' and b'' belong to $P^*(\subset K)$ and $[a'', b''] \subset \mathbf{Q} \cup \operatorname{int} \mathbf{Q} (= \mathrm{Q}) \subset D$. Since K is relatively convex in D, $[a', b'] \subset [a'', b''] \subset K \subset K'$ (see Figure).



There remains the case where $[a', b']$ is exterior to $\mathbf{Q}$. Take a point t in the open interval (a', b') and draw the ray $\mathrm{R}(t \to c)$. The initial segment $[t, c]$ is exterior to $\mathbf{Q}$, but interior to $[a', b', c] \subset D$. At c the ray enters $\operatorname{int} \mathbf{Q}$ $(\subset D)$ and stays there until it hits P^* at some point t'. Thus t sees $t'(\in K)$ via D, hence $t \in K'$ (see Figure).



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Applying Theorem 2.4 to the singleton $K = \{x\}$ $(x \in D)$ we obtain

Corollary 2.5: *Let D be a simply connected set in $\Re^2$. If $x \in D$ then $st(x, D)$ is relatively convex in D.*

Continuing by induction on n, we obtain in the same way:

Corollary 2.6: *Let D be a simply connected set in $\Re^2$, and x a point in D. Then $st_n(x, D)$ is relatively convex in D.*

This corollary is an extension of Lemma 1 in [BB].

2.2 The Monotonicity Lemma. Let $D \subset \Re^2$ and $a, b \in D$ be such that $\rho_D(a, b) = n$. Let $\mathrm{L}(a, b)$ be a polygonal path in D that connects a with b and has only n edges. As we have already mentioned, $\mathrm{L}(a, b)$ is simple. We order $\mathrm{L}(a, b)$ in a natural way from a to b. For $s, t \in \mathrm{L}(a, b)$, $s \neq t$, we write $s < t$ to denote that s precedes t on $\mathrm{L}(a, b)$.

Assume $\mathrm{L}(a, b) = \langle p_0, p_1, \ldots, p_n \rangle$, where $p_0 = a$ and $p_n = b$.

The main result of this section, Lemma 2.8, which will be needed in [MP2], states that if D is simply connected, then the function $\rho_D(a, s)$ $(s \in \mathrm{L}(a, b))$ is monotone nondecreasing.

Lemma 2.7: *If $s \in [p_i, p_{i+1}] \subset \mathrm{L}(a, b)$, then $\rho_D(a, s) = i$ or $i + 1$.*

Proof: Immediate from the minimality of $\mathrm{L}(a, b)$ and the fact that $\rho_D(a, p_j) = j$ $(j = 0, 1, \ldots, n)$. ∎

Lemma 2.8: *Let D be a simply connected set in $\Re^2$, and let a, b be two points in D such that $\rho_D(a, b) = n$. Then the function $\rho_D(a, s)$ is a nondecreasing integer valued function on $\mathrm{L}(a, b)$.*

Proof: Assume that $s, t \in \mathrm{L}(a, b)$ and $s < t$. We show that $\rho_D(a, s) \leq \rho_D(a, t)$. Suppose that $s \in [p_i, p_{i+1}]$ and $t \in [p_j, p_{j+1}]$. Clearly $i \leq j$, since $s < t$. If $i < j$ then $\rho_D(a, s) \leq \rho_D(a, t)$ by Lemma 2.7. Assume, therefore, that $i = j$. If $\rho_D(a, s) > \rho_D(a, t)$, then, again by Lemma 2.7, $\rho_D(a, t) = i$ $(= \rho_D(a, p_i))$ and $\rho_D(a, s) = i + 1$. But then $s \in [p_i, t]$, with $t, p_i \in st_i(x, D)$. By Corollary 2.6, $s \in st_i(x, D)$ as well, i.e., $\rho_D(a, s) \leq i$, contradicting our assumption. ∎

3. Helly's Topological Theorem for relatively convex subsets

3.1 Introduction. Helly's Topological Theorem in $\Re^2$ (see [H]) states that a family of compact, connected and simply connected sets in the plane has nonempty intersection, provided every three members intersect, and every two intersect in a connected set.

In this section we give an elementary proof of this statement for the case where all members of the family are relatively convex and polygonally connected subsets of a simply connected set $D \subset \Re^2$. In view of Lemma 3.2 below, we can dispense with the requirement that the intersection of each two sets be connected.

Theorem 3.1: *Let D be a simply connected set in $\Re^2$, and let $\{K_i : i \in I\}$ with $|I| \geq 3$ be a family of subsets of D that are compact, polygonally connected and relatively convex in D, and such that each three have a point in common. Then the intersection of the whole family is nonempty.*

The proof of Theorem 3.1 is based on the following two lemmas, which hold for a simply connected set D in the plane.

Lemma 3.2: *Let $K_1, K_2 \subset D$ be two polygonally connected and relatively convex subsets of D. If $K_1 \cap K_2 \neq \emptyset$ then $K_1 \cap K_2$ is also polygonally connected and relatively convex in D.*

Note that Lemma 3.2 does not require the sets K_i to be closed.

Lemma 3.3: *If $K_1, K_2, K_3, K_4 \subset D$ are compact, polygonally connected and relatively convex in D, and the intersection of every three K_i's is nonempty, then $K_1 \cap K_2 \cap K_3 \cap K_4 \neq \emptyset$.*

3.2 Proof of Lemma 3.2. It is clear that $K_1 \cap K_2$ is relatively convex in D. We only have to show that it is polygonally connected. Let a, c be two points in $K_1 \cap K_2$. We denote by k_1 the combinatorial distance from a to c via K_1, that is $k_1 = \rho_{K_1}(a, c)$. Similarly we define $k_2 = \rho_{K_2}(a, c)$ and $k_{12} = \rho_{K_1 \cap K_2}(a, c)$.

If $a = c$ then $k_1 = k_2 = k_{12} = 0$, and there is nothing to prove.

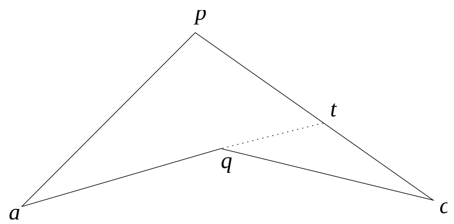
If $a \neq c$ and $[a, c] \subset D$ then $k_1 = k_2 = k_{12} = 1$.

If $[a, c]$ is not in D then $k_1 \geq 2$, $k_2 \geq 2$ and $k_{12} \geq 2$. In this case, we shall prove two auxiliary lemmas, and Lemma 3.2 will follow as an immediate consequence.

Lemma 3.4: *If $k_1 = k_2 = 2$ then $k_{12} = 2$.*

Proof: Let $\langle a, p, c\rangle$ be a two-edge path joining a and c via K_1, and let $\langle a, q, c\rangle$ be a two-edge path joining a and c via K_2.

If these two paths meet only at the endpoints, then their union is a simple closed quadrilateral $\mathbf{Q}$. Since the segment $[a, c]$ is not in D, it must be an exterior diagonal of $\mathbf{Q}$.

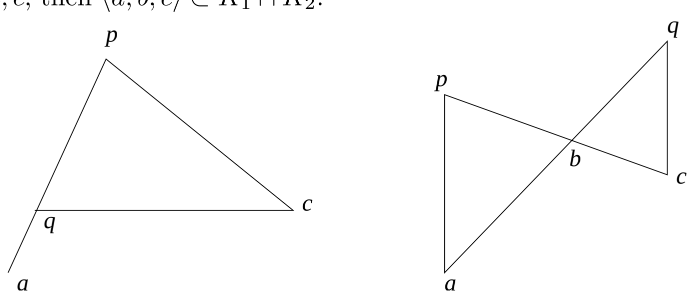


Assume, without loss of generality, that the interior angle of $\mathbf{Q}$ at q is greater than π.

Extending the segment $[a, q]$ beyond q inside Q we reach a point t in $[p, c] \subset K_1$, as pictured in the figure above. Since D is simply connected, it includes the interior of Q. In particular, $[a, t] \subset D$. Both a and t are in K_1, and K_1 is relatively convex in D, hence $[a, q] \subset K_1$. In particular, $q \in K_1$. Therefore, $\langle a, q, c\rangle \subset K_1 \cap K_2$.

If $\langle a, p, c\rangle$ and $\langle a, q, c\rangle$ meet at one of the interior vertices p, q, say $q \in \langle a, p, c\rangle$, then $\langle a, q, c\rangle \subset K_1 \cap K_2$.

If $q \notin \langle a, p, c\rangle$ and $p \notin \langle a, q, c\rangle$, and the two paths meet at a point b other than a, c, then $\langle a, b, c\rangle \subset K_1 \cap K_2$.



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Lemma 3.5: *If $\min(k_1, k_2) \geq 2$, then $k_{12} \leq k_1 + k_2 - 2$.*

Proof: By induction on $k_1 + k_2$.

If $k_1 + k_2 = 4$ then $k_1 = k_2 = 2$ and our claim follows from Lemma 3.4. Assume, therefore, that $k_1 + k_2 \geq 5$

Let $P_1 = \langle a = p_0, p_1, \ldots, p_{k_1} = c\rangle \subset K_1$ and $P_2 = \langle a = q_0, q_1, \ldots, q_{k_2} = c\rangle \subset K_2$ be polygonal paths joining a and c in K_1 and K_2 respectively. From the minimality of k_1 and k_2 we infer that both these paths are simple. They may or may not share a point other than a, c.

Suppose there exists a point $b \in P_1 \cap P_2$ different from a and c. If $b \in (a, p_1) \cap (a, q_1)$, then the segments $[a, p_1]$ and $[a, q_1]$ overlap, and we replace b by p_1 or q_1, whichever comes first. We make the same stipulation at the other end of the paths.

Define:

$$k_1' = \rho_{K_1}(a,b), \quad k_1'' = \rho_{K_1}(b,c), \quad k_2' = \rho_{K_2}(a,b), \quad k_2'' = \rho_{K_2}(b,c)$$
$$k_{12}' = \rho_{K_1 \bigcap K_2}(a,b), \quad k_{12}'' = \rho_{K_1 \bigcap K_2}(b,c).$$

Clearly, $k_i \le k_i' + k_i'' \le k_i + 1$ for $i = 1, 2$.

We shall consider the following cases.

Case I: $\min(k_1', k_2') \ge 2$ and $\min(k_1'', k_2'') \ge 2$. Because of the restrictions that we have just imposed on the point b, at least one of the inequalities $k_1'' \le k_1$, $k_2'' \le k_2$ is strict, and therefore $k_1'' + k_2'' < k_1 + k_2$. Similarly for k_i': $k_1' + k_2' < k_1 + k_2$. This enables us to apply the inductive hypothesis to the pairs a, b and b, c, to obtain;

$$k_{12}' \le k_1' + k_2' - 2$$
$$k_{12}'' \le k_1'' + k_2'' - 2$$

Since $k_{12} \le k_{12}' + k_{12}''$, we conclude that $k_{12} \le k_{12}' + k_{12}'' \le k_1' + k_2' - 2 + k_1'' + k_2'' - 2 \le k_1 + 1 + k_2 + 1 - 2 - 2 = k_1 + k_2 - 2$, and our claim holds.

Case II: $\min(k_1', k_2') = 1$ and $\min(k_1'', k_2'') \ge 2$.

If $\min(k_1', k_2') = 1$ then $k_1' = k_2' = k_{12}' = 1$. Applying the inductive hypothesis to the pair b, c, we obtain;

$$k_{12}'' \le k_1'' + k_2'' - 2 < k_1 + k_2 - 2;$$

hence $k_{12} \le k_{12}' + k_{12}'' = 1 + k_{12}'' < 1 + k_1 + k_2 - 2$, i.e., $k_{12} \le k_1 + k_2 - 2$, as claimed. The same argument holds for the case where $\min(k_1', k_2') \ge 2$ and $\min(k_1'', k_2'') = 1$.

Case III: $\min(k_1', k_2') = 1$ and $\min(k_1'', k_2'') = 1$.

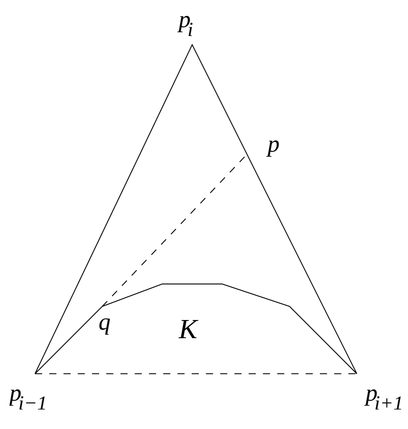
In this case $k_1' = k_2' = k_{12}' = k_1'' = k_2'' = k_{12}'' = 1$, hence $k_{12} \le k_{12}' + k_{12}'' = 2$ $(\le k_1 + k_2 - 2)$. ∎

Now assume that $P_1 \cap P_2 = \{a, c\}$. In this case $P = P_1 \cup P_2$ is a simple closed polygon. The sum of the interior angles of a simple closed n-gon in the plane is $(n-2) \cdot 180°$. It follows that at least three vertices of P have an interior angle smaller than $180°$. Thus, P has an interior angle smaller than $180°$ at some vertex other than a, c.

Assume, without loss of generality, that this vertex is $p_i \in P_1$, for some $0 < i < k_1$. Consider the triangle $\Delta = [p_{i-1}, p_i, p_{i+1}]$. If P does not meet $\operatorname{int}\Delta$, then $\operatorname{int}\Delta \subset \operatorname{int} P$, hence $\Delta = cl(\operatorname{int}\Delta) \subset cl(\operatorname{int} P) = P \bigcup \operatorname{int} P \subset D$. In particular we find that p_{i-1} and p_{i+1} see each other via D (and hence via K_1) which contradicts the minimality of P_1. Therefore, P does meet $\operatorname{int}\Delta$.

If an edge $e = [s, t]$ of P meets $\operatorname{int}\Delta$, then either both endpoints s, t of e are in $\operatorname{int}\Delta$, or one endpoint is in $\operatorname{int}\Delta$, and e meets the "base" $[p_{i-1}, p_{i+1}]$ of Δ. It follows that $P \cap \operatorname{int}\Delta \subset \operatorname{conv}(\{p_{i-1}, p_{i+1}\} \cup (\operatorname{vert} P \cap \operatorname{int}\Delta))$. The polygon $K = \operatorname{conv}(\{p_{i-1}, p_{i+1}\} \cup (\operatorname{vert} P \cap \operatorname{int}\Delta))$ meets the boundary of Δ in $[p_{i-1}, p_{i+1}]$ only. Note that $\operatorname{int}\Delta - K \subset \operatorname{int} P$. Each vertex of K, except p_{i-1} and p_{i+1}, sees p_i via $\operatorname{int} P \subset D$, and therefore cannot be a vertex of (the minimal path) P_1. It follows that any vertex of K, other than p_{i-1} and p_{i+1}, must be an interior vertex of the path P_2.

Denote by q the vertex of K adjacent to p_{i-1} in $\operatorname{int}\Delta$, and extend the segment $[p_{i-1}, q]$ beyond q, into $\operatorname{int}\Delta - K$, until it hits the edge $[p_i, p_{i+1}] (\subset K_1)$ at a point p. The segment $[p_{i-1}, p] = [p_{i-1}, q] \cup [q, p]$ is in D, hence in K_1. Replace the edges $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$ of P_1 by $[p_{i-1}, p]$ and $[p, p_{i+1}]$. This leads to another minimal path P_1' from a to c in K_1. P_1' does meet P_2 at the point q, which is interior to both paths. This situation has been treated before.



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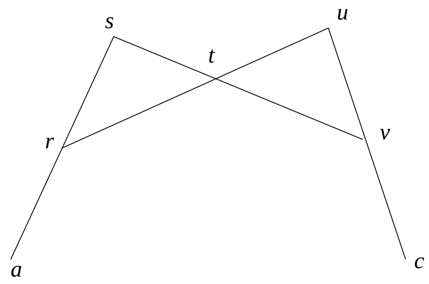
Note: The bound $k_1 + k_2 - 2$ for k_{12} is tight.

To obtain an example with $k_1 = 3 + \alpha \geq 3$ and $k_2 = 3 + \beta \geq 3$, consider a convex polygon $\mathbf{Q}$ with vertices $a, p_0, p_1, \ldots, p_\beta, b, q_\alpha, \ldots, q_1, q_0, c$ (in this order on the boundary of Q), and with acute interior angles at the vertices a, c. Denote by P the polygonal path $\langle a, p_0, p_1, \ldots, p_\beta, b, q_\alpha, \ldots, q_1, q_0, c\rangle$. Extend the edge $[a, p_0]$ beyond p_0, and the edge $[b, q_\alpha]$ beyond b until they meet at a point s. Extend the edge $[c, q_0]$ beyond q_0, and the edge $[p_\beta, b]$ beyond b until they meet at a point t.

Let Δ_1 and Δ_2 be the (usually nonconvex) polygons bounded by $\langle s, p_0, \ldots, p_\beta, b, s\rangle$ and by $\langle t, q_0, \ldots, q_\alpha, b, t\rangle$, respectively, and define $K_1 = P \cup \Delta_1$ and $K_2 = P \cup \Delta_2$. Thus $K_1 \cap K_2 = P$, $\rho_{K_1}(a, c) = 3 + \alpha$, $\rho_{K_2}(a, c) = 3 + \beta$ and $\rho_P(a, c) = \alpha + \beta + 4 = (3 + \alpha) + (3 + \beta) - 2$.

The following figure shows the construction for $\alpha = 2$ and $\beta = 3$.



3.3 Proof of Lemma 3.3.

Lemma 3.6: *Let D be a simply connected subset of $\Re^2$, and let K be a relatively convex subset of D. If P is a simple closed polygon in K, then* $\operatorname{int} P \subset K$.

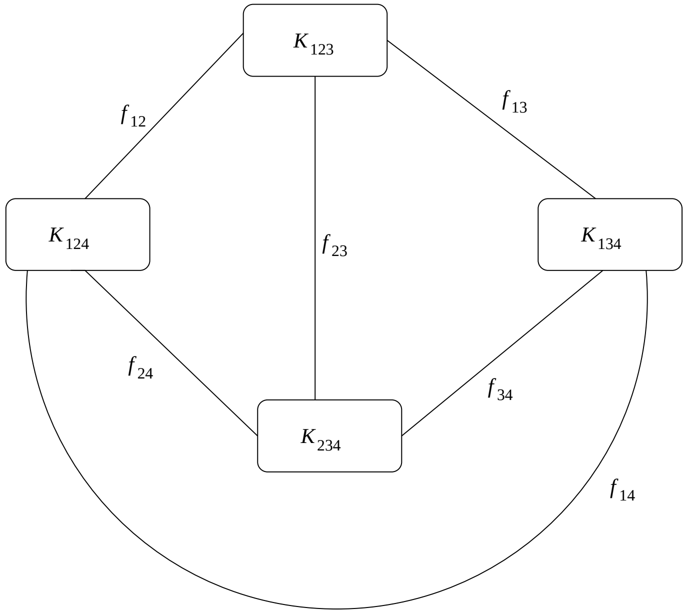
Proof: Since D is simply connected, $\operatorname{int} P \subset D$. Let x be a point in $\operatorname{int} P$, and let $L \subset \Re^2$ be a line through x. x divides L into two rays, say L_+ and L_-, and both rays meet P. Let $x_+ (x_-)$ be the first point of $L_+ (L_-)$ in P. Then $x_+, x_- \in K$, $(x_+, x_-) \subset \operatorname{int} P \subset D$, hence $[x_+, x_-] \subset K$. In particular, $x \in [x_+, x_-] \subset K$. ∎

Definition 3.7: Let S, T be subsets of $\Re^2$. An (S, T)-path is a simple polygonal path $L = \langle s = l_0, l_1, \ldots, l_k = t\rangle$ that starts at some point $s \in S$, ends at a point $t \in T$, and does not meet $S \cup T$ in any other point.

Lemma 3.8: *Suppose $K \subset \Re^2$ is polygonally connected, and S, T are two disjoint nonempty compact subsets of K. Then there is an (S, T)-path in K.*

Proof: Choose points $s_0 \in S$ and $t_0 \in T$. There is a simple polygonal path L' that joins s_0 with t_0 in K. Let s be the last point of L' in S. Let t be the first point of the part of L' from s to t_0 that lies in T. Finally, let L be the part of L' from s to t. ∎

Proof of Lemma 3.3:



Assume, on the contrary, that $K_1 \cap K_2 \cap K_3 \cap K_4 = \emptyset$. Let us denote the intersection $K_i \cap K_j \cap K_k$ $(i, j, k \in \{1, 2, 3, 4\})$ by K_{ijk}. By our assumptions, K_{123}, K_{124}, K_{134} and K_{234} are four pairwise disjoint nonempty compact subsets of D. By Lemma 3.2, they are also polygonally connected, and relatively convex in D. We shall construct now a simple polygonal embedding of the complete graph on four vertices into D, with the following specifications:

VERTICES: $a_{ijk} \in K_{ijk}$ $(\{i, j, k\} \subset \{1, 2, 3, 4\}$ and $|\{i, j, k\}| = 3)$. The order of the indices is not important. Thus $a_{123} = a_{231} = a_{321}$, etc.

EDGES: e_{ij} $(=e_{ji})$ between a_{ijk} and a_{ijl} $(\{i, j, k, l\} = \{1, 2, 3, 4\})$. The "edge" e_{ij} is a simple polygonal path in $K_i \cap K_j$ with endpoints a_{ijk} and a_{ijl}. Distinct edges with a common endpoint share this endpoint only. Distinct edges without a common endpoint are disjoint.

CONSTRUCTION:

STEP I: For $1 \le i < j \le 4$ let f_{ij} $(= f_{ji})$ be a (K_{ijk}, K_{ijl})-path within $K_i \cap K_j$ $(\{i, j, k, l\} = \{1, 2, 3, 4\})$. The existence of f_{ij} is guaranteed by Lemma 3.8.

The six paths f_{ij} are pairwise disjoint, except possibly for common endpoints. Moreover, the interior points of f_{ij} do not belong to any of the four sets K_{123}, K_{124}, K_{134} and K_{234}. To prove this fact, take, for example, a point z interior to f_{12}. Then $z \in K_1 \cap K_2$, but $z \notin K_3$ and $z \notin K_4$. Thus z cannot belong to any other path f_{ij}, nor to any of the sets $K_{\alpha\beta\gamma}$. (See the figure above).

STEP II: For each of the four sets K_{ijk} we extend the three paths f_{ij}, f_{ik} and f_{jk} that emanate from K_{ijk}, back into K_{ijk}, until they meet in a common point a_{ijk}. To be specific, consider the set K_{123}. Denote by b_{ij} the endpoint of f_{ij} in K_{123} ($ij = 12$ or 23 or 13). We will distinguish between the following three situations:

1. $b_{12} = b_{23} = b_{13}$. Call this common point a_{123} and do nothing else.
2. Two of the endpoints coincide and the third one is different. Assume, e.g., $b_{12} = b_{13} \neq b_{23}$. Define $a_{123} = b_{12} = b_{13}$, connect a_{123} to b_{23} by a simple polygonal path in K_{123}, and attach this path to f_{23}.
3. The three endpoints b_{ij} are distinct. Connect b_{12} and b_{13} by a simple polygonal path g in K_{123}. If $b_{23} \in g$, define $a_{123} = b_{23}$ and attach the part of g from b_{12} to a_{123} to f_{12}, and the other part (from b_{13} to a_{123}) to f_{13}. If $b_{23} \notin g$, choose a $(\{b_{23}\}, g)$-path h in K_{123}. Denote by a_{123} the point where h hits g. Attach h to f_{23}, and the two parts of g determined by a_{123} to f_{12} and f_{13}, respectively.

The graph we have drawn defines a planar map with 4 vertices, 6 edges and 4 faces ($v - e + f = 2$, by Euler's formula). The boundary of each face is a circuit with at least three edges. Since the number of edge-face incidences is 12 ($= 2e$), each face has exactly three edges (and three vertices). This applies, in particular, to the unbounded face. Assume, e.g., that the boundary of the unbounded face has vertices a_{123}, a_{124} and a_{134} joined by the edges e_{12}, e_{14} and e_{13}. The union of these three edges (actually paths) is a simple closed polygon T in K_1. The unbounded face of our map is just the exterior of T. The remaining vertex a_{234} is not there, hence $a_{234} \in \operatorname{int} T$ ($\subset K_1$ by Lemma 3.6). Thus $a_{234} \in K_1 \cap K_2 \cap K_3 \cap K_4$, contrary to our assumption. ∎

3.4 PROOF OF THEOREM 3.1. The sets K_i are compact, so in order to show that the intersection of the whole family is nonempty, it suffices to show that every finite subfamily has a nonempty intersection.

Let $K_1, \ldots, K_n$ ($n \geq 3$) be any n elements of the family, such that each three share a point. If $n = 3$ there is nothing to prove. If $n = 4$, they all meet by Lemma 3.3. If $n \geq 5$, we proceed by induction. Define sets $K'_1, \ldots, K'_{n-1}$ as follows: $K_i' = K_i$ for $1 \leq n \leq n-2$ and $K_{n-1}' = K_{n-1} \cap K_n$. Clearly, all these sets are compact, polygonally connected (Lemma 3.2), relatively convex in D, and every three of them meet (Lemma 3.3). Thus, by the inductive hypothesis,

$$\bigcap_{i=1}^{n-1} K_i' \neq \emptyset.$$

But

$$\bigcap_{i=1}^{n-1} K_i' = \bigcap_{i=1}^{n} K_i,$$

and our claim holds. ∎

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